PROBLEM 1:

"GEOMETRIC UNSHARPNESS" TYPICALLY REFERS TO THE PENUMBRA AND EDGE BLURRING EFFECTS

a) GEOMETRIC UNSHARPNESS IS DECREASED (THAT IS, THE IMAGE IS LESS BLURRY). THE BREAST TISSUE IS EFFECTIVELY FLATTENED IN THE Z-DIRECTION, REDUCING EDGE BLURRING OF FEATURES.

b) IMAGE CONTRAST WILL BE IMPROVED. REDUCED BLURRING IMPROVES CONTRAST, REDUCED SCATTERING (DUE TO REDUCED x-ray PATH LENGTHS THROUGH TISSUE) IMPROVES CONTRAST, THE DYNAMIC RANGE REQUIREMENT IS REDUCED (POTENTIALLY IMPROVING CONTRAST), AND A SOFTER x-ray SPECTRUM (i.e., LOWER ENERGIES) COULD BE USED, FURTHER IMPROVING CONTRAST.

c) THE REQUIRED x-ray DOSE IS LOWER. THE x-ray PATH LENGTH IS REDUCED, SO A LOWER INTENSITY BEAM CAN BE USED FOR AN EQUIVALENT NUMBER OF PHOTONS REACHING THE DETECTOR.

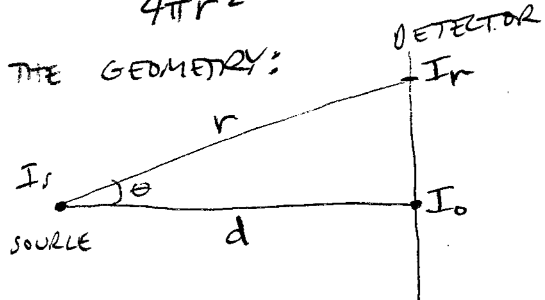
NOTE: THE BOOK IS SLOPPY IN IT'S NOTATION. HERE THEY USE I_0 FOR THE INTENSITY AT THE ORIGIN OF THE DETECTOR AND I_S FOR THE SOURCE INTENSITY.

PROBLEM 2:

a) LET I_S BE THE INTENSITY AT THE SOURCE. THE INVERSE SQUARE LAW MEANS THAT, AT A DISTANCE r FROM THE SOURCE, THE INTENSITY I_r IS GIVEN BY:

$$I_r = \frac{I_S}{4\pi r^2}$$

GIVEN THE GEOMETRY:



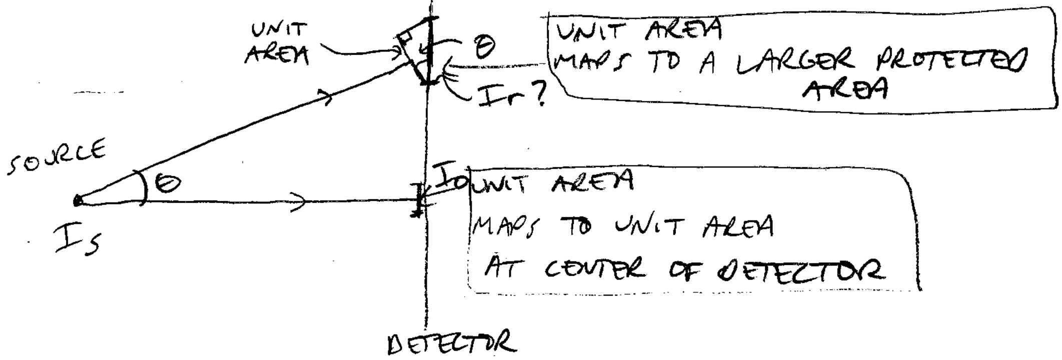
WE SEE THAT: $d = r\cos\theta$ OR $\cos\theta = \frac{d}{r}$

AND: $I_0 = \frac{I_S}{4\pi d^2}$ ⇐ FROM INVERSE SQUARE LAW.

SO WE CAN WRITE:

$$I_r = \frac{I_0}{4\pi r^2} = \frac{\cancel{4\pi} d^2 I_0}{\cancel{4\pi} r^2} = I_0 \frac{d^2}{r^2} = \boxed{I_0 \cos^2\theta}$$ ⇐ FROM INVERSE SQUARE LAW

b) FROM OBLIQUITY, WE SEE THAT A UNIT AREA ORTHOGONAL TO THE DIRECTION OF X-ray PROPAGATION MAPS TO A LARGER AREA ON THE DETECTOR AS WE MOVE AWAY FROM THE CENTER OF THE DETECTOR.



FROM THE GEOMETRY, WE SEE THAT THE PROJECTED UNIT AREA HAS AREA $\frac{1}{\cos\theta}$:

$$A_{PROJECTED} = \frac{A_{ORTHOGONAL}}{\cos\theta}$$

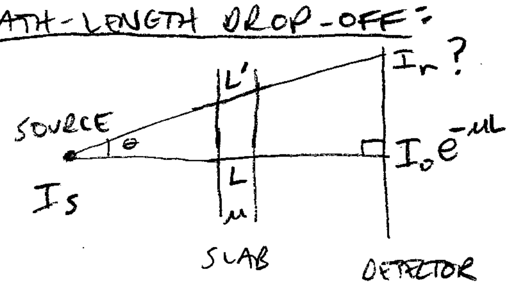
INTENSITY IS PROPORTIONAL TO THE NUMBER OF PHOTONS ARRIVING PER UNIT AREA PER UNIT TIME. SINCE OUR AREA <u>INCREASED</u> BY A FACTOR OF $\frac{1}{\cos\theta}$, INTENSITY MUST <u>DROP</u> BY A FACTOR $\cos\theta$. (AREA IS IN THE DENOMINATOR FOR INTENSITY.)

SO:

$$\boxed{I_r = I_o \cos\theta}$$ ⟸ FROM OBLIQUITY

<u>NOTE:</u> ONCE AGAIN, THE BOOK IS SLOPPY WITH NOTATION. THEY NOW SWITCH WHAT THEY USED TO CALL I_r TO I_D. I'M KEEPING IT I_r.

c) <u>PATH-LENGTH DROP-OFF:</u>



PATH LENGTH THROUGH MATERIAL IS INCREASED:

$$L' = \frac{L}{\cos\theta}$$

IF THE MATERIAL HAS LINEAR ATTENUATION COEFFICIENT μ, THEN AT THE DETECTOR ORIGIN WE HAVE:

$$I_{ORIGIN} = I_0 e^{-\mu L}$$

I_0 IS THE INTENSITY THAT WOULD BE PRESENT AT THE ORIGIN <u>WITHOUT</u> THE SLAB PRESENT. IT IS DIFFERENT FROM I_S BECAUSE OF INVERSE SQUARE DROP-OFF.

AND I_r IS GIVEN BY:

$$I_r = I_0 e^{-\frac{\mu L}{\cos\theta}}$$ (IGNORING OTHER FACTORS)

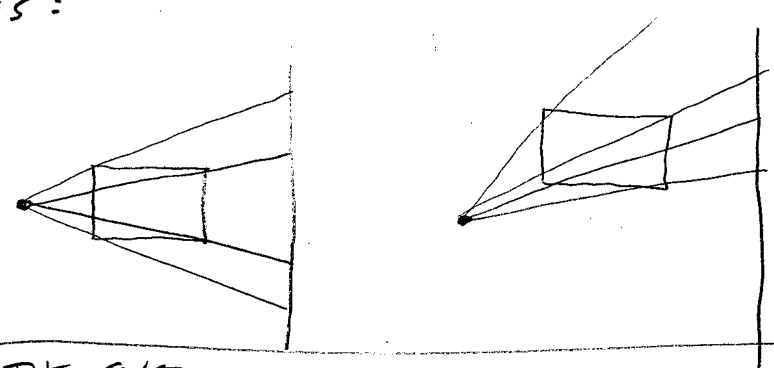
PUTTING IT ALL TOGETHER, WE HAVE:

$$I_r = I_0 \underbrace{\cos^2\theta}_{\text{INVERSE SQUARE DROP-OFF}} \underbrace{\cos\theta}_{\text{OBLIQUITY}} \underbrace{e^{-\frac{\mu L}{\cos\theta}}}_{\text{PATH-LENGTH DROP-OFF}}$$

$$\boxed{I_r = I_0 \cos^3\theta \, e^{-\frac{\mu L}{\cos\theta}}}$$ ✓

5.1) THE SYSTEM <u>IS NOT</u> SHIFT INVARIANT IN THE z DIRECTION BECAUSE OBJECT MAGNIFICATION CHANGES AS AN OBJECT MOVES IN z DUE TO THE DIVERGENCE OF THE x-rays.

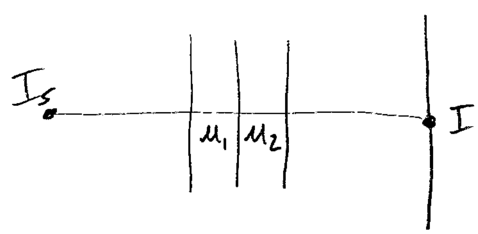
FOR OBJECT SHIFTS IN x AND y, THE SYSTEM IS ONLY SHIFT INVARIANT IF THE OBJECT IS INFINITESIMALLY THIN IN z. OTHERWISE, AREAS OF THE OBJECT AT DIFFERENT DEPTHS EXPERIENCE DIFFERENT SHIFTS AND EDGE BLURRING CHANGES:



SO: $\boxed{\text{THE SYSTEM } \underline{\text{IS NOT}} \text{ SHIFT INVARIANT IN GENERAL.}}$

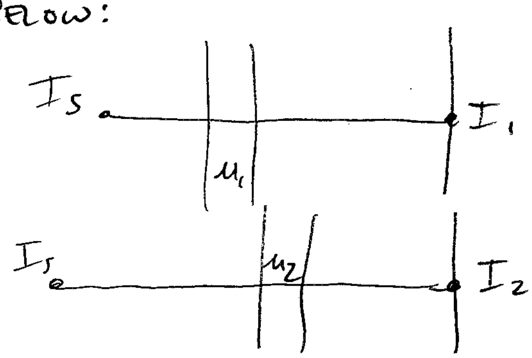
<u>IS IT LINEAR?</u>

IF THE SYSTEM IS LINEAR, THEN WHEN WE IMAGE TWO SLABS AS SHOWN BELOW:



WE SHOULD HAVE: $I = I_1 + I_2$

WHERE I_1 AND I_2 ARE THE INTENSITIES AT THE DETECTOR IMAGING THE SLABS INDIVIDUALLY AS SHOWN BELOW:



IS THIS TRUE?

ASSUMING A MONOENERGETIC X-ray BEAM AND SLAB THICKNESSES OF Δx, WE HAVE:

$$I = I_0 e^{-(\mu_1 \Delta x + \mu_2 \Delta x)} = I_0 e^{-\mu_1 \Delta x} e^{-\mu_2 \Delta x}$$

AND:

$$I_1 = I_0 e^{-\mu_1 \Delta x}$$

$$I_2 = I_0 e^{-\mu_2 \Delta x}$$

DOES $I = I_1 + I_2$?

$$I_1 + I_2 = I_0 e^{-\mu_1 \Delta x} + I_0 e^{-\mu_2 \Delta x}$$

$$= I_0 \left(e^{-\mu_1 \Delta x} + e^{-\mu_2 \Delta x} \right) \neq I \,!$$

SO $\boxed{\text{NOT LINEAR}}$.

5.2) a) THE HIGHEST ENERGY IS DETERMINED BY THE PEAK X-ray TUBE VOLTAGE. IF THE PEAK VOLTAGE IS 50 kV, THE PEAK ENERGY OF AN X-ray PHOTON WILL BE 50 keV.

THE ENERGY SPECTRUM IS DETERMINED BY SEVERAL FACTORS:

1. IT WILL BE ZERO ABOVE THE PEAK PHOTON ENERGY.
2. IT WILL INCLUDE CHARACTERISTIC X-ray SPIKES AT CERTAIN ENERGIES THAT DEPEND ON THE ATOMIC MAKE-UP OF THE ANODE OF THE X-ray TUBE.
3. IT WILL INCLUDE A CONTINUOUS SPECTRUM OF BREMSSTRAHLUNG X-rays WITH A TYPICAL SHAPE UP TO THE PEAK ENERGY.
4. IT WILL BE ATTENUATED BY PHOTON ABSORPTION IN THE ANODE ITSELF (THE "ANODE HEEL EFFECT") AND OTHER PHOTON ABSORPTION IN THE TUBE.

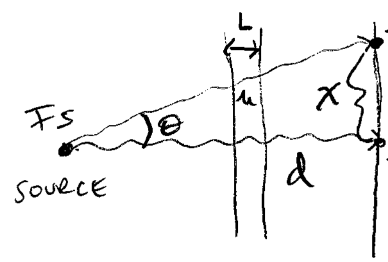
b) LOW-ENERGY PHOTONS ARE UNDESIRABLE BECAUSE THEY ARE USUALLY COMPLETELY ABSORBED BY THE BODY, CONTRIBUTING TO DOSE BUT NOT IMAGE QUALITY.

MEASURES THAT CAN BE TAKEN TO REDUCE THE NUMBER OF LOW-ENERGY PHOTONS ENTERING THE BODY INCLUDE:

1. RESTRICTION $\Rightarrow$ THAT IS, REDUCING THE APERTURE SIZE TO LIMIT THE AREA EXPOSED
2. FILTERING $\Rightarrow$ MATERIALS LIKE ALUMINUM MAKE GOOD FREQUENCY-SELECTIVE FILTERS FOR x-rays, EFFECTIVELY ATTENUATING LOWER-ENERGY PHOTONS MORE THAN HIGHER-ENERGY PHOTONS.

c) "BEAM HARDENING" IS THE INCREASING OF AN x-ray BEAM'S EFFECTIVE ENERGY AS IT PROPAGATES THROUGH TISSUES OR MATERIALS. IT OCCURS BECAUSE MOST MATERIALS HAVE LARGER ATTENUATION COEFFICIENTS AT LOWER x-ray ENERGIES.

5.6)



$$I_x = I_0 \cos^3\theta \, e^{-\mu L/\cos\theta}$$

$$I_c = I_0 e^{-\mu L}$$

WE WANT:

$$I_x = 0.95 I_c \quad \text{OR} \quad \frac{I_x}{I_c} = 0.95$$

SO:

$$\frac{I_x}{I_c} = \frac{\cancel{I_0}\cos^3\theta \, e^{-\left(\frac{\mu L}{\cos\theta}\right)}}{\cancel{I_0} e^{-\mu L}} \approx \frac{\cos^3\theta \, \cancel{e^{-\mu L}}}{\cancel{e^{-\mu L}}}$$

$\frac{\mu L}{\cos\theta} \approx \mu L$ FOR SMALL θ

SO: $$0.95 = \frac{I_x}{I_c} \approx \cos^3\theta$$

$$\cos^3\theta \approx 0.95$$

$$\cos\theta \approx 0.983$$

$$\theta \approx 10.56^\circ$$

$d = 2\,\text{m}$

SO x IN OUR DIAGRAM IS:

$$x = d\tan\theta$$

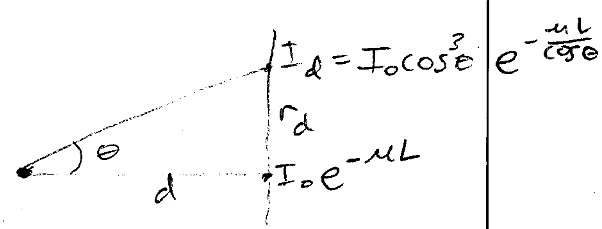
$$x = 2\tan(10.56^\circ) = 0.373\ \text{m}$$

SO THE IMAGE CAN HAVE A MAXIMAL RADIUS OF 0.373 m OR A DIAMETER OF 0.746 m.

5.8)

a) THE WEIGHTING NEEDS TO COMPENSATE FOR THE $\cos^3\theta$ DEPENDENCE FROM OBLIQUITY AND INVERSE-SQUARE DROP-OFF AND THE PATH LENGTH FACTOR $e^{-\frac{\mu L}{\cos\theta}}$. IMAGE INTENSITY IS GIVEN BY:

$$I_d = I_0\cos^3\theta\, e^{-\frac{\mu L}{\cos\theta}}$$



WHERE:

$$\cos\theta = \frac{d}{\sqrt{d^2 + r_d^2}}$$

WE WANT TO CORRECT I_d TO $I_d' = I_0 e^{-\mu L}$.

IF WE DIVIDE I_d BY $I_0\cos^3\theta$ WE HAVE:

$$\frac{I_d}{I_0\cos^3\theta} = \frac{\cancel{I_0\cos^3\theta}\, e^{-\frac{\mu L}{\cos\theta}}}{\cancel{I_0\cos^3\theta}} = e^{\frac{-\mu L}{\cos\theta}}$$

TO GET RID OF THE $\frac{1}{\cos\theta}$ IN THE EXPONENT, LET'S RAISE BOTH SIDES TO A POWER $\cos\theta$:

$$\left(\frac{I_d}{I_0\cos^3\theta}\right)^{\cos\theta} = \left(e^{-\frac{\mu L}{\cos\theta}}\right)^{\cos\theta} = e^{-\mu L}$$

WE'RE ALMOST THERE. IF WE MULTIPLY BOTH SIDES BY I_o, WE HAVE OUR DESIRED I_d':

$$I_o\left(\frac{I_d}{I_o\cos^3\theta}\right)^{\cos\theta} = I_o e^{-\mu L} = I_d'$$

SO:

$$I_d' = I_o\left(\frac{I_d}{I_o\cos^3\theta}\right)^{\cos\theta} = I_d \cdot \underbrace{\frac{I_o}{I_d}\left(\frac{I_d}{I_o\cos^3\theta}\right)^{\cos\theta}}_{}$$

SO THIS IS OUR WEIGHTING FUNCTION AS A FUNCTION OF $\cos\theta$

SO:

$$\text{WEIGHTING} = \frac{I_o}{I_d}\left(\frac{I_d}{I_o\cos^3\theta}\right)^{\cos\theta},$$

WHERE

$$\cos\theta = \frac{d}{\sqrt{d^2+r_d^2}}$$

WE CAN NOW SUBSTITUTE IN:

$$I_d = I_o\cos^3\theta\, e^{\frac{-\mu L}{\cos\theta}}$$

$$\text{WEIGHTING} = \frac{\cancel{I_o}}{I_o\cos^3\theta\, e^{-\frac{\mu L}{\cos\theta}}}\left(\frac{\cancel{I_o\cos^3\theta}\, e^{-\frac{\mu L}{\cos\theta}}}{\cancel{I_o\cos^3\theta}}\right)^{\cos\theta}$$

$$= \frac{e^{-\mu L}}{\cos^3\theta\, e^{-\frac{\mu L}{\cos\theta}}}$$

$$= \frac{e^{-\mu L}\, e^{\frac{\mu L}{\cos\theta}}}{\cos^3\theta}$$

$$\boxed{\text{WEIGHTING} = \frac{e^{\mu L\left(\frac{1}{\cos\theta}-1\right)}}{\cos^3\theta} \quad \text{WHERE} \quad \cos\theta = \frac{d}{\sqrt{d^2+r_d^2}}}$$

NOTE THAT THIS EXPRESSION IMPLICITLY ASSUMES THAT μ ONLY VARIES IN THE z DIRECTION.

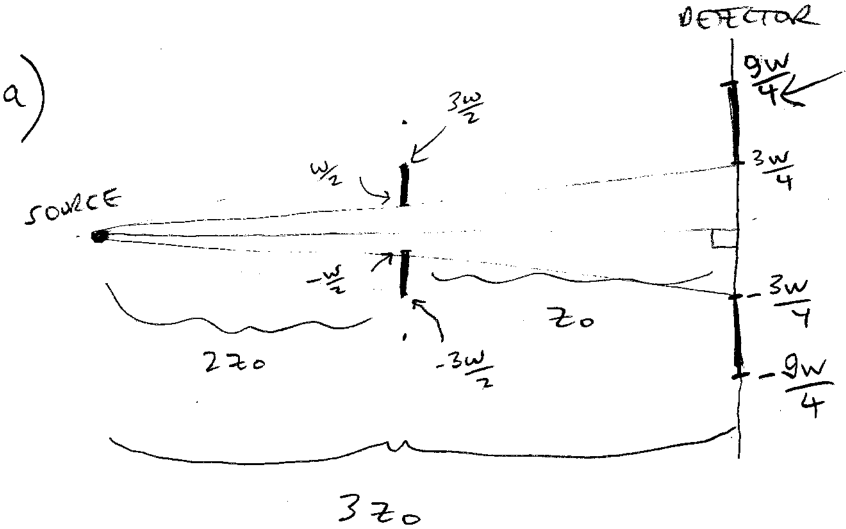
b) I'LL ACCEPT SEVERAL ANSWERS HERE. FIRST, IMAGE QUALITY COULD BE SAID TO IMPROVE BECAUSE THE IMAGE HAS REMOVED INTENSITY DROP-OFF FROM GEOMETRIC AND PATH LENGTH VARIATIONS, AND IS NOW TRUER TO REALITY.

SECOND, IF YOU CONSIDER A TUMOR WITH INTENSITY I_t IN THE CENTER OF THE DETECTOR SURROUNDED BY A BRIGHTER TISSUE WITH INTENSITY $I_b > I_t$, THEN YOU COULD SAY CONTRAST IS IMPROVED BECAUSE BRIGHTNESS DROP-OFF IN THE BACKGROUND TISSUE I_b WILL BE RESTORED. HOWEVER, THE CONTRAST REALLY MATTERS MORE RIGHT AT THE BOUNDARY OF THE TUMOR, AND THINGS AREN'T IMPROVED RIGHT AT THE BOUNDARY BECAUSE THE CORRECTION FACTOR IS VIRTUALLY THE SAME FOR BOTH TISSUES.

WHAT ABOUT SNR? WELL, WE'RE REALLY ENHANCING BOTH SIGNAL AND NOISE IN ANY GIVEN VOXEL BY THE SAME FACTOR, SO THE SNR OF ANY GIVEN VOXEL WON'T CHANGE. BUT YOU COULD ARGUE CNR IS IMPROVED, DEPENDING ON WHERE YOU DEFINE AND MEASURE SIGNAL AVERAGES AND NOISE STANDARD DEVIATION.

THIS PROBLEM SHOULD GET YOU THINKING. IF IT DID AND YOU MADE A WELL-REASONED ARGUMENT, YOU'LL GET FULL CREDIT.

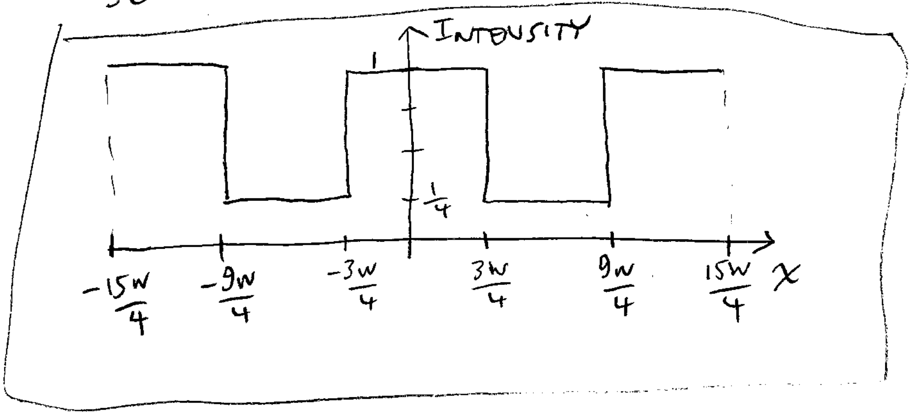
5.23) a)



MAGNIFICATION IS $\frac{3z_0}{2z_0} = \frac{3}{2}$

IF INTENSITY = 1 AT DETECTOR ORIGIN, THEN THE INTENSITY IS .25 OR $\frac{1}{4}$ AT THE DARK BARS.

SO:



b) IF THE DARK BARS ARE TARGETS, WE HAVE:

$$C = \frac{\frac{1}{4} - 1}{1} = \boxed{-\frac{3}{4}}$$

c) $h(x,y) = \text{sinc}(\alpha x)\,\text{sinc}(\beta y) \quad , \alpha, \beta > 0$

SO:

$$H(u,v) = \mathcal{F}_{2D}\{h(x,y)\} = \frac{1}{\alpha}\text{rect}\left(\frac{u}{\alpha}\right) \cdot \frac{1}{\beta}\text{rect}\left(\frac{v}{\beta}\right)$$

IDEAL LOW-PASS FILTER BAND-LIMITS OUR OUTPUT!

THE HIGHEST SPATIAL FREQUENCIES THAT WILL BE PASSED BY $H(u,v)$ ARE $U_0 = \frac{\alpha}{2}$ AND $V_0 = \frac{\beta}{2}$.

THIS IS BECAUSE $H(u,v) = \frac{1}{\alpha}\text{rect}\left(\frac{u}{\alpha}\right)\cdot\frac{1}{\beta}\text{rect}\left(\frac{v}{\beta}\right)$

EXTENDS FROM $\frac{u}{\alpha} = -\frac{1}{2}$ TO $\frac{u}{\alpha} = \frac{1}{2}$

AND $\frac{v}{\beta} = -\frac{1}{2}$ TO $\frac{v}{\beta} = \frac{1}{2}$ (DEFINITION OF THE RECT)

NYQUIST THEN TELLS US THAT THE LARGEST Δx AND Δy WE CAN HAVE ARE:

$$\Delta x_{MAX} = \frac{1}{2U_0} = \frac{1}{\cancel{2}\cdot\frac{\alpha}{\cancel{2}}} = \boxed{\frac{1}{\alpha} = \Delta x_{max}}$$

$$\Delta y_{max} = \frac{1}{2V_0} = \frac{1}{\cancel{2}\cdot\frac{\beta}{\cancel{2}}} = \boxed{\frac{1}{\beta} = \Delta y_{max}}$$

d) $\mu(E) = \ln\left(\frac{640\text{ keV}}{E}\right)\ \text{cm}^{-1}$

$$\mu(160\text{ keV}) = \ln\left(\frac{640\text{ keV}}{160\text{ keV}}\right)\text{cm}^{-1} = \ln(4)\ \text{cm}^{-1}$$

NOW:

$$I_d = I_0 e^{-\mu\Delta x}$$

$$\frac{I_d}{I_0} = \frac{1}{4} = e^{-\mu\Delta x} = e^{-\ln(4)\Delta x}$$

$$e^{-\ln(4)\Delta x} = \frac{1}{4}$$

$$-\ln(4)\Delta x = \ln\left(\frac{1}{4}\right)$$

$$-\ln(4)\Delta x = -\ln(4)$$

$$\Delta x = \frac{-\ln(4)}{-\ln(4)} = 1\ \text{cm.}$$

CM, SINCE μ IS IN cm^{-1}!

$$\boxed{\Delta x = 1\ \text{cm.}}$$