PROBLEM 1:

a) $M_{xy} = M_0 \sin\alpha\, e^{-t/T_2} = M_0 e^{-t/T_2}$ ($\alpha = 90°$)

$M_z = M_0 + [M_0 \cos\alpha - M_0] e^{-t/T_1} = M_0(1 - e^{-t/T_1})$ ($\cos\alpha = 0$)

$$|M| = \sqrt{M_{xy}^2 + M_z^2} = \sqrt{M_0^2 e^{-2t/T_2} + M_0^2(1-e^{-t/T_1})^2}$$

$$\boxed{|M| = M_0 \sqrt{e^{-2t/T_2} + 1 - 2e^{-t/T_1} + e^{-2t/T_1}}}$$

b) IF $T_2 < T_1$, THEN $e^{-t/T_2} < e^{-t/T_1}$

ALSO, SINCE e^{-t/T_2} AND e^{-t/T_1} ARE ALWAYS LESS THAN 1 FOR POSITIVE t, WE KNOW THAT:

$$e^{-2t/T_2} < e^{-t/T_2} < e^{-t/T_1}$$

AND:

$$e^{-2t/T_1} < e^{-t/T_1}$$

NOW LOOK AT OUR EXPRESSION FOR $|M|$:

$$|M| = M_0 \underbrace{\sqrt{e^{-2t/T_2} + 1 - 2e^{-t/T_1} + e^{-2t/T_1}}}_{\text{IF THIS IS LESS THAN 1, THEN } |M| < M_0 \text{!}}$$

SO, WE WANT TO SHOW:

$$\sqrt{e^{-2t/T_2} + 1 - 2e^{-t/T_1} + e^{-2t/T_1}} < 1$$

$$e^{-2t/T_2} + 1 - 2e^{-t/T_1} + e^{-2t/T_1} < 1$$

$$e^{-2t/T_2} + e^{-2t/T_1} - 2e^{-t/T_1} < 0$$

$$\underbrace{e^{-2t/T_2}}_{< e^{-t/T_1}} + \underbrace{e^{-2t/T_1}}_{< e^{-t/T_1}} < 2e^{-t/T_1} \quad ✓$$

PROBLEM 2:

a) IF WE WAIT A TIME t_d AFTER OUR PULSE, WE HAVE:

TISSUE A: $M_{xy_A} = M_0 e^{-t_d/T_{2A}}$

TISSUE B: $M_{xy_B} = M_0 e^{-t_d/T_{2B}}$

AND:

$$CNR = \frac{\left| M_0 e^{-t_d/T_{2A}} - M_0 e^{-t_d/T_{2B}} \right|}{\sigma_N}$$

TO MAXIMIZE CNR:

$$\frac{d\, CNR}{d t_d} = 0 \Rightarrow \frac{M_0}{\sigma_N}\left[-\frac{1}{T_{2A}} e^{-\frac{t_d}{T_{2A}}} + \frac{1}{T_{2B}} e^{-\frac{t_d}{T_{2B}}} \right] = 0$$

$$\frac{e^{-\frac{t_d}{T_{2B}}}}{T_{2B}} = \frac{e^{-\frac{t_d}{T_{2A}}}}{T_{2A}}$$

$$\frac{-t_d}{T_{2B}} - \ln(T_{2B}) = \frac{-t_d}{T_{2A}} - \ln(T_{2A})$$

$$\frac{t_d}{T_{2A}} - \frac{t_d}{T_{2B}} = \ln(T_{2B}) - \ln(T_{2A})$$

$$t_d\left(\frac{1}{T_{2A}} - \frac{1}{T_{2B}}\right) = \ln\left(\frac{T_{2B}}{T_{2A}}\right)$$

$$t_d = \frac{\ln\left(\frac{T_{2B}}{T_{2A}}\right)}{\frac{1}{T_{2A}} - \frac{1}{T_{2B}}} = \frac{T_{2A} T_{2B} \ln\left(\frac{T_{2B}}{T_{2A}}\right)}{T_{2B} - T_{2A}}$$

$$\boxed{t_d = \frac{T_{2A} T_{2B} \ln\left(\frac{T_{2B}}{T_{2A}}\right)}{T_{2B} - T_{2A}}}$$

b) IN THIS CASE, IMMEDIATELY BEFORE THE SECOND TIP WE HAVE:

TISSUE A: $M_{ZA} = M_0\left(1 - e^{-\frac{t_{prep}}{T_{1A}}}\right)$

TISSUE B: $M_{ZB} = M_0\left(1 - e^{-\frac{t_{prep}}{T_{1B}}}\right)$

IMMEDIATELY AFTER THE SECOND TIP, $M_z \rightarrow M_{xy}$ AND $M_{xy} \rightarrow -M_z$, SO:

AFTER 2^{nd} TIP:

$$M_{xyA} = M_0\left(1 - e^{-\frac{t_{prep}}{T_{1A}}}\right)$$

$$M_{xyB} = M_0\left(1 - e^{-\frac{t_{prep}}{T_{1B}}}\right)$$

$$CNR = \frac{\left|M_0\left(1 - e^{-\frac{t_{prep}}{T_{1A}}}\right) - M_0\left(1 - e^{-\frac{t_{prep}}{T_{1B}}}\right)\right|}{\sigma_N}$$

$$CNR = \frac{\left|M_0\left(e^{-\frac{t_{prep}}{T_{1B}}} - e^{-\frac{t_{prep}}{T_{1A}}}\right)\right|}{\sigma_N}$$

TO MAXIMIZE CNR:

$$\frac{d\,CNR}{d\,t_{prep}} = 0 \qquad \frac{M_0}{\sigma_N}\left(-\frac{e^{-\frac{t_{prep}}{T_{1B}}}}{T_{1B}} + \frac{e^{-\frac{t_{prep}}{T_{1A}}}}{T_{1A}}\right) = 0$$

SIMILAR TO BEFORE:

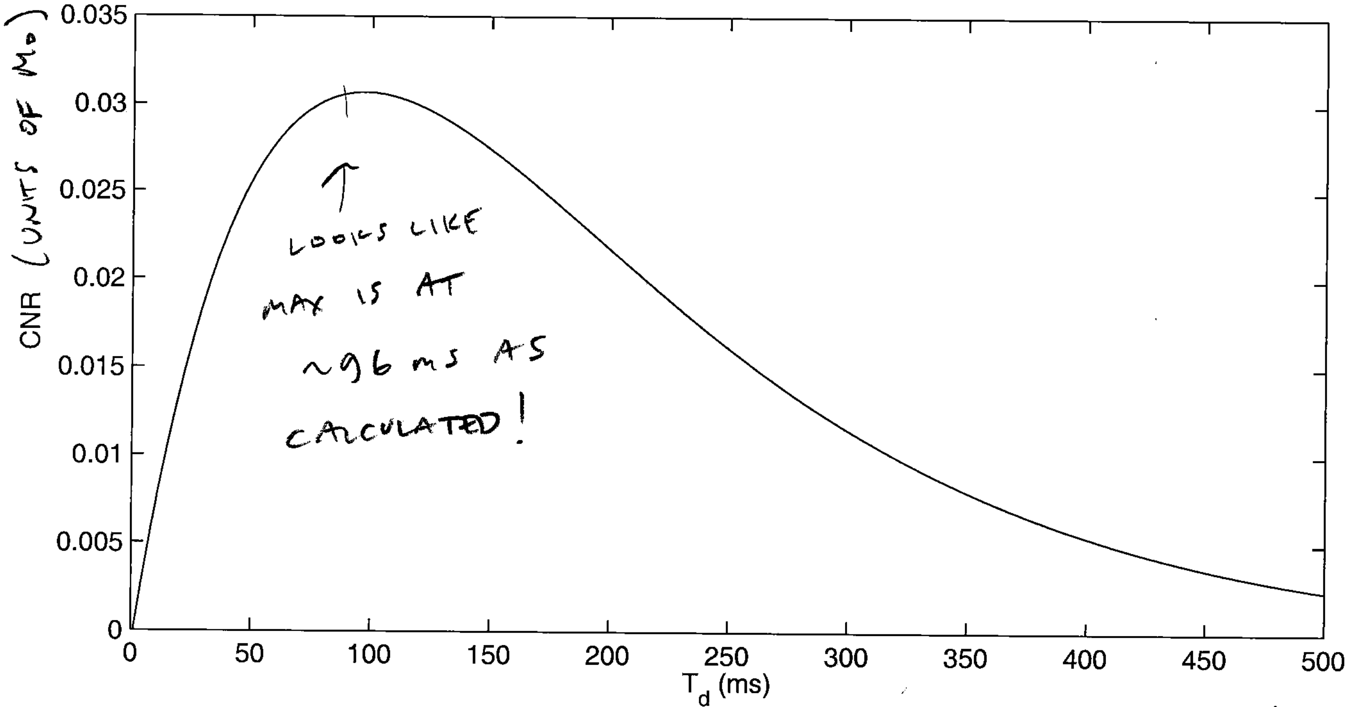
$$\frac{e^{-\frac{t_{prep}}{T_{1A}}}}{T_{1A}} = \frac{e^{-\frac{t_{prep}}{T_{1B}}}}{T_{1B}}$$

$$\boxed{t_{prep} = \frac{T_{1A}T_{1B}\ln\left(\frac{T_{1A}}{T_{1B}}\right)}{T_{1A} - T_{1B}}}$$

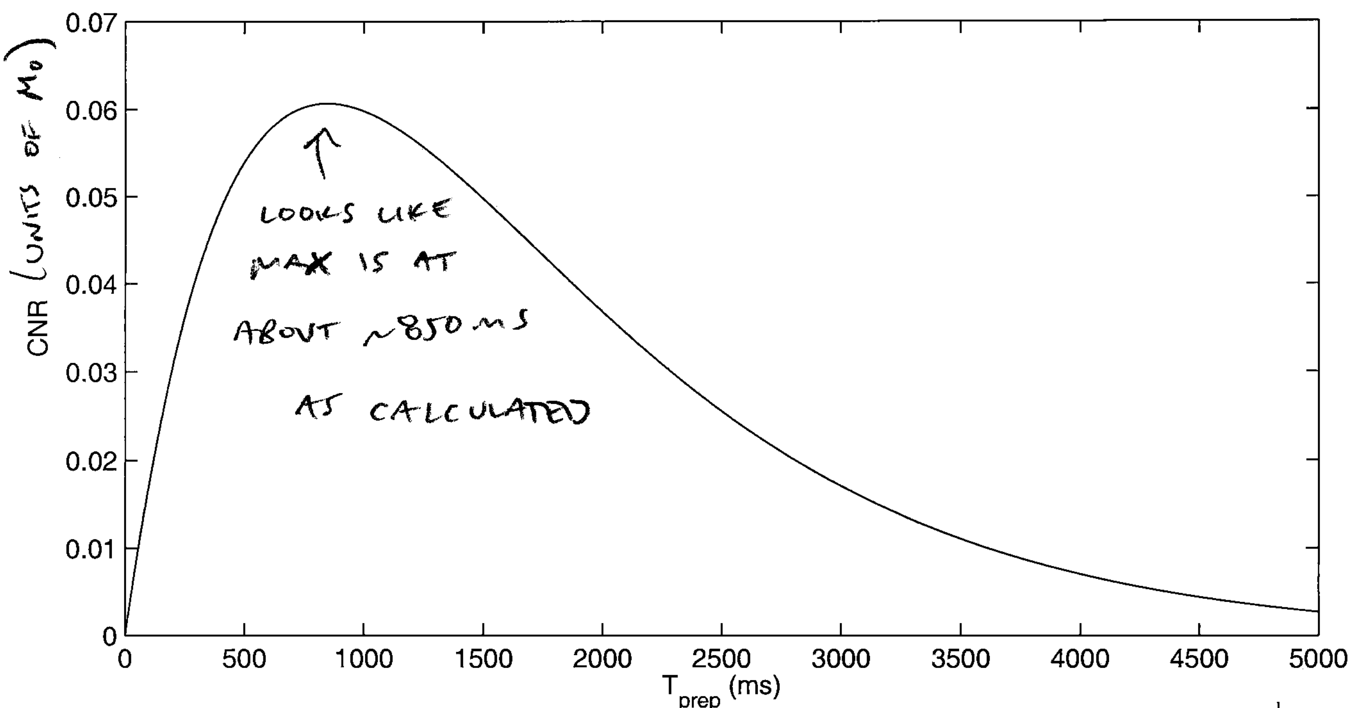
c) $$t_d = \frac{(92)(100)\ln\left(\frac{100}{92}\right)}{100 - 92} = \boxed{95.9 \text{ ms} = t_d}$$

$$t_{prep} = \frac{(780)(920)\ln\left(\frac{780}{920}\right)}{780 - 920} = \boxed{846 \text{ ms} = t_{prep}}$$

d)


CNR (UNITS OF M₀)
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
0
50
100
150
200
250
300
350
400
450
500
T_d (ms)
LOOKS LIKE MAX IS AT ~96 ms AS CALCULATED!

e)


CNR (UNITS OF M₀)
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000
T_prep (ms)
LOOKS LIKE MAX IS AT ABOUT ~850 ms AS CALCULATED

f) WE GET BETTER CNR WITH THE SECOND SCHEME (DOING TWO TIPS FOR T_1 CONTRAST)

g) IMMEDIATELY AFTER OUR SECOND TIP, WE HAVE (FROM PART (b)):

$$M_{xyA} = M_0\left(1 - e^{-\frac{t_{prep}}{T_{1A}}}\right)$$
$$M_{xyB} = M_0\left(1 - e^{-\frac{t_{prep}}{T_{1B}}}\right)$$

IF WE NOW WAIT A TIME t_d, WE WILL SIMPLY HAVE t_2 DECAY:

$$M_{xyA} = M_0\left(1 - e^{-\frac{t_{prep}}{T_{1A}}}\right) e^{-\frac{t_d}{T_{2A}}}$$
$$M_{xyB} = M_0\left(1 - e^{-\frac{t_{prep}}{T_{1B}}}\right) e^{-\frac{t_d}{T_{2B}}}$$

SO:

$$CNR = \frac{\left| M_0\left(1 - e^{-\frac{t_{prep}}{T_{1A}}}\right) e^{-\frac{t_d}{T_{2A}}} - M_0\left(1 - e^{-\frac{t_{prep}}{T_{1B}}}\right) e^{-\frac{t_d}{T_{2B}}} \right|}{\sigma_N}$$

$$\boxed{CNR = \frac{M_0}{\sigma_N}\left| \left(1 - e^{-\frac{t_{prep}}{T_{1A}}}\right) e^{-\frac{t_d}{T_{2A}}} - \left(1 - e^{-\frac{t_{prep}}{T_{1B}}}\right) e^{-\frac{t_d}{T_{2B}}} \right|}$$

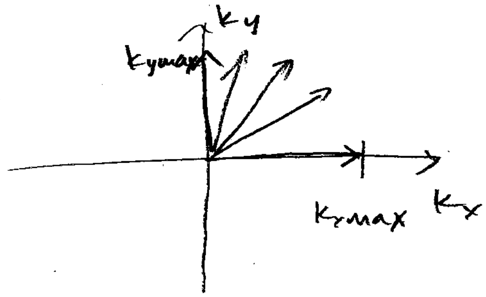
h)

```
for td = 0:500
for tprep = 0:5000
CNR(td+1,tprep+1)=abs((1-exp(-tprep/780))*exp(-td/92) - (1-exp(-tprep/920))*exp(-td/100));
end
end

max(CNR(:))
ans = 0.0607

max(CNRb)
ans = 0.0607
```

← NO! A QUICK CALCULATION W/ MATLAB SHOWS WE GET SAME MAX AS W/ SCHEME IN PART (b)!

PROBLEM 3:

a) WE'LL ACHIEVE OUR HIGHEST RESOLUTION BY SHOOTING STRAIGHT OUT FROM THE CENTER OF K-SPACE FOR 8 ms, MAXING OUT OUR GRADIENTS. (THIS IS 2DPR!)



$G_{xmax} = 4 \ \frac{G}{cm} = 4\times10^{-4} \ \frac{T}{cm}$

WHAT IS k_{xmax}?

$$k_{xmax} = \frac{\gamma}{2\pi} G_{xmax} \cdot 8 \text{ ms}$$

$$k_{xmax} = \left(42.575\times10^{6} \ \frac{Hz}{T}\right)\left(4\times10^{-4} \ \frac{T}{cm}\right)\left(8\times10^{-3} \ s\right)$$

$$k_{xmax} = 136.24 \ cm^{-1}$$

$$\delta_x = \frac{1}{2(136.24)} \ cm = 0.00367 \ cm = 0.0367 \ mm$$

$$\boxed{\delta_x = 36.7 \ \mu m}$$

SIMILARLY:

$$k_{ymax} = \left(42.575\times10^{6} \ \frac{Hz}{T}\right)\left(3\times10^{-4} \ \frac{T}{cm}\right)\left(8\times10^{-3} \ s\right)$$

$$= 102.18 \ cm^{-1}$$

$$\delta_y = \frac{1}{2(102.18)} \ cm = \boxed{48.9 \ \mu m = \delta_y}$$

b) SAMPLING EVERY 4 μs FOR 8 ms YIELD $\frac{8 \ ms}{4 \ \mu s} = 2000$ SAMPLES, SO:

$$\Delta k_x = \frac{k_{xmax}}{2000} = \frac{136.24}{2000} \ cm^{-1} = 0.06812 \ cm^{-1}$$

$$\boxed{FOV_x = 14.7 \ cm} = \frac{1}{\Delta k_x}$$

$$\Delta k_y = \frac{k_{ymax}}{2000} = \frac{102.18 \ cm^{-1}}{2000} = 0.05109 \ cm^{-1}$$

$$\boxed{FOV_y = 19.6 \ cm}$$

c) $k_{xmax} = \left(11.262 \times 10^6 \frac{Hz}{T}\right)\left(4 \times 10^{-4} \frac{T}{cm}\right)\left(8 \times 10^{-3} s\right)$

$k_{xmax} = 36.04 \ cm^{-1}$

$\delta_x = \frac{1}{2(36.04)} \ cm = $ **$139 \ \mu m = \delta_x$**

$k_{ymax} = (11.262 \times 10^6)(3 \times 10^{-4})(8 \times 10^{-3}) = 27.03 \ cm^{-1}$

$\delta_y = 185 \ \mu m$

PROBLEM 4:

AFTER OUR 1ST TIP:

$M_{xy}(1) = M_0$ (AFTER 1ST TIP)

$M_z(1) = 0$

RIGHT BEFORE OUR SECOND TIP: ⟸ w/ TR = 40 ms

$M_{xy} = M_0 e^{-TR/T_2} = M_{xy}(1) e^{-TR/T_2}$

$M_z = M_0 + [M_z(1) - M_0] e^{-TR/T_1}$

RIGHT AFTER SECOND TIP:

$M_{xy}(2) = M_0 + [M_z(1) - M_0] e^{-TR/T_1}$

$M_z(2) = -M_{xy}(1) e^{-TR/T_2}$

NEGATIVE OF M_{xy} JUST BEFORE TIP!

IN GENERAL:

$M_{xy}(k) = M_0 + [M_z(k-1) - M_0] e^{-TR/T_1}$

$M_z(k) = -M_{xy}(k-1) e^{-TR/T_2}$

TR = 40 ms

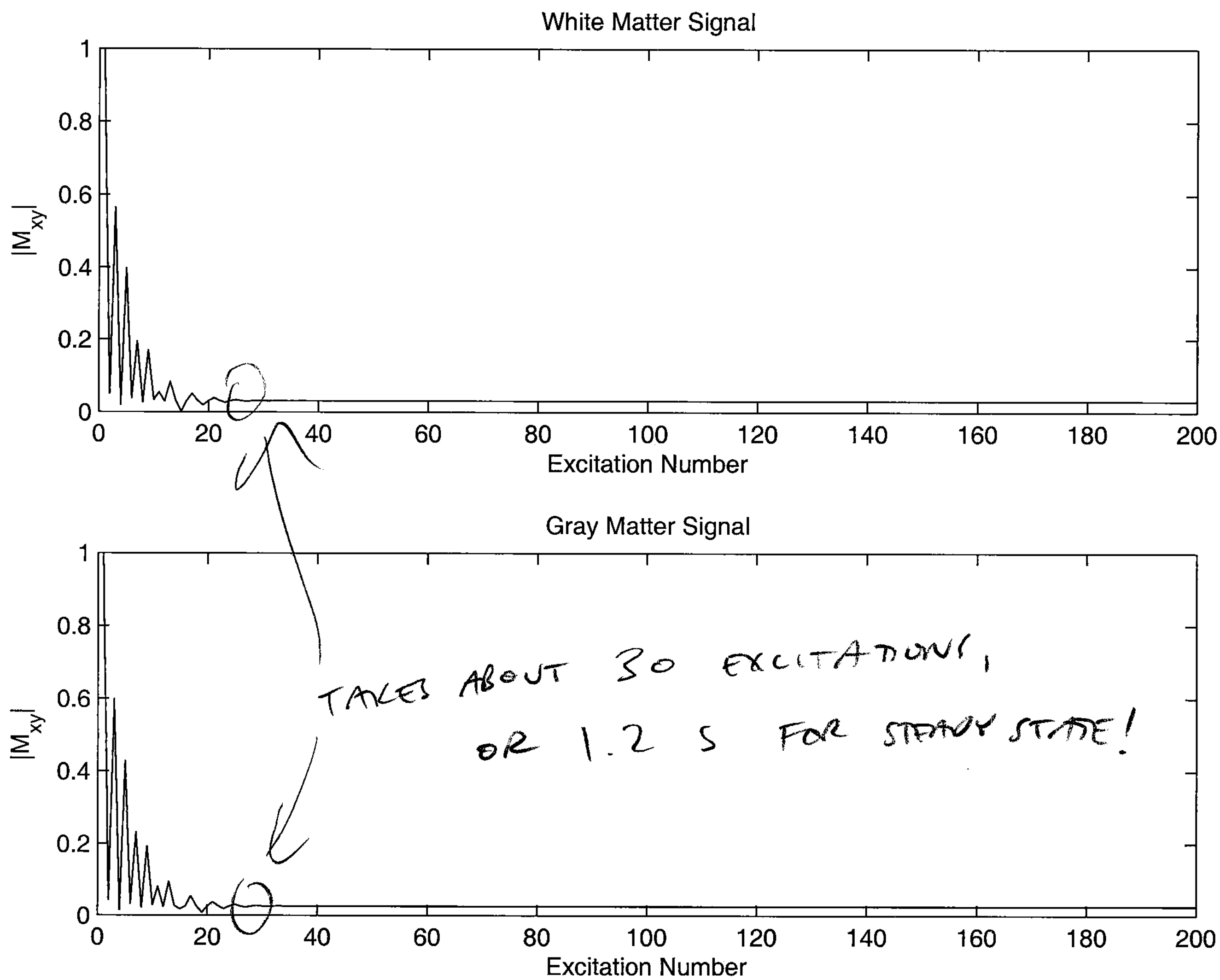
SO LET'S CODE THAT IN MATLAB FOR:

$k = 1:200$

AND: $M_{xy}(1) = M_0 = 1$ FOR BOTH GRAY AND WHITE MATTER.

$M_z(1) = 0$

```
% Put in T1 and T2 of Gray Matter and White Matter
T1_WM = 780;
T1_GM = 920;
T2_WM = 92;
T2_GM = 100;

% Our time increment between tips is 40 ms
TR = 40;

% Set up arrays that are 200 long to hold Mxy and Mz for GM and WM
Mxy_GM = zeros(200,1);
Mxy_WM = zeros(200,1);
Mz_GM = zeros(200,1);
Mz_WM = zeros(200,1);

% Set up initial conditions (in other words, Mxy values after 1st tip
Mxy_GM(1) = 1;
Mxy_WM(1) = 1;
Mz_GM(1) = 0;  % Just for good measure :)
Mz_WM(1) = 0;

% Now run the loop and compute what happens after excitations 2 to 200
for k = 2:200
    Mxy_GM(k) = 1 + (Mz_GM(k-1) - 1)*exp(-TR/T1_GM);
    Mxy_WM(k) = 1 + (Mz_WM(k-1) - 1)*exp(-TR/T1_WM);
    Mz_GM(k) = -Mxy_GM(k-1)*exp(-TR/T2_GM);
    Mz_WM(k) = -Mxy_WM(k-1)*exp(-TR/T2_WM);
end

% And plot the Mxy values (our signal)
k = 1:200;

figure;
subplot(2,1,1);
plot(abs(Mxy_WM));
title('White Matter Signal');
xlabel('Excitation Number');
ylabel('|M_x_y|');

subplot(2,1,2);
plot(abs(Mxy_GM));
title('Gray Matter Signal');
xlabel('Excitation Number');
ylabel('|M_x_y|');
```


White Matter Signal
$|M_{xy}|$
Excitation Number
Gray Matter Signal
$|M_{xy}|$
Excitation Number
TAKES ABOUT 30 EXCITATIONS,
OR 1.2 S FOR STEADY STATE!