- GRADED HW 3 AND HW 4 AVAILABLE, SOLUTIONS POSTED ON WEB
- HW 5 GRADED (+ SOLUTIONS) BY EOD MONDAY
- MIDTERM REVIEW SESSION MONDAY 5-6 pm
- MIDTERM REVIEW/ INFO SHEET AVAILABLE ON WEB SITE BY EOD FRIDAY
- ADVANCED LECTURE NEXT WEEK, MONDAY 4-5 pm
- MIDTERM IN TESTING CENTER NEXT WEEK, TUESDAY–FRIDAY (3 HOUR, CLOSED NOTES, CLOSED BOOK, MIXED MULTIPLE CHOICE AND HOMEWORK-TYPE PROBLEMS)

PARALLEL-RAY RECONSTRUCTION IN CT

RECALL THAT FOR MULTI-ENERGETIC BEAMS, THE INTENSITY I_d AT A GIVEN DETECTOR IS GIVEN BY:

LOCATION OF DETECTOR

$$I_d = \int_0^{E_{max}} \underbrace{E\,S(E)}_{\text{x-ray spectrum}}\, e^{-\int_0^{r(x,y)} \mu(s,E)\,ds}\, dE$$

← BASIC X-RAY IMAGING EQUATION

DEALING WITH THE INTEGRAL OVER E IS A PAIN. WE TYPICALLY JUST DEAL WITH AN EFFECTIVE ENERGY $\bar{E}$. WE THEN HAVE:

$$I_d = I_0\, e^{-\int_0^{r(x,y)} \mu(s)\,ds}$$

INSTEAD OF DEALING WITH INTENSITIES, WE DEFINE THE BASIC MEASUREMENT OF A CT SCANNER AS:

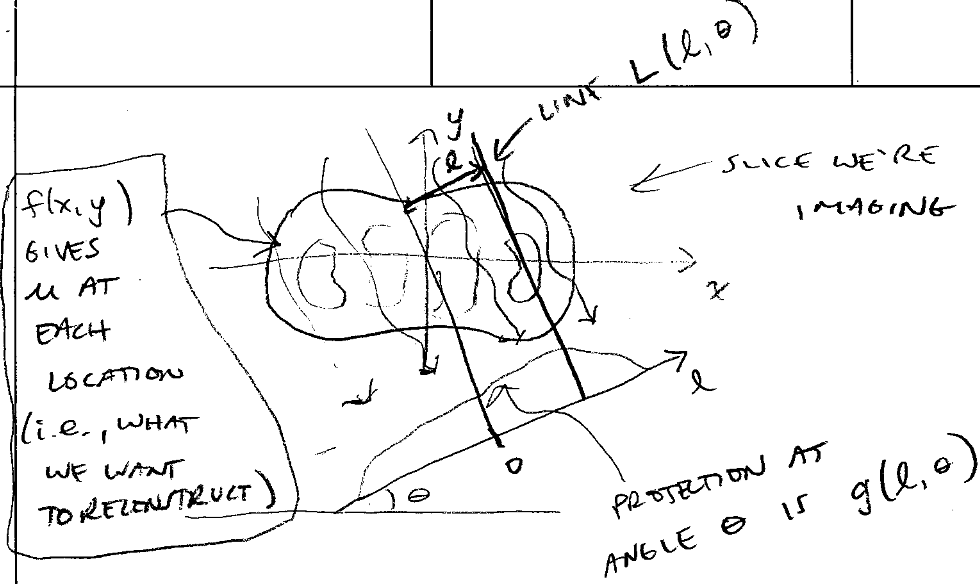
$$\boxed{g_D \equiv -\ln\left(\frac{I_d}{I_0}\right) = \int_0^{r(x,y)} \mu(s)\,ds}$$

THUS, THE BASIC MEASUREMENT OF A CT SCANNER IS JUST A LINE INTEGRAL OF THE LINEAR ATTENUATION COEFFICIENT AT THE EFFECTIVE ENERGY OF THE SCANNER.

FROM THIS WE RECONSTRUCT CT NUMBER AT EACH PIXEL.

$$\boxed{h = 1000 \times \frac{\mu - \mu_{H_2O}}{\mu_{H_2O}}}$$ IN HOUNSFIELD UNITS

NOW LET'S PUT SOME MATH BEHIND OUR PROJECTIONS.



AT A GIVEN ANGLE θ, $g(\ell,\theta)$ IS GIVEN BY:

$$\boxed{g(\ell,\theta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\delta(x\cos\theta+y\sin\theta-\ell)\,dx\,dy}$$

- $g(\ell,\theta)$ AT A GIVEN θ IS A PROJECTION.
- $g(\ell,\theta)$ FOR ALL ℓ AND θ IS THE 2-D RADON TRANSFORM OF $f(x,y)$.
- A PLOT OF $g(\ell,\theta)$ WITH θ ALONG THE VERTICAL AXIS AND ℓ ALONG THE HORIZONTAL AXIS IS CALLED A SINOGRAM

EXAMPLE: (6.2 IN BOOK)

LET: $f(x,y)=\begin{cases}1 & , x^2+y^2\le 1\\ 0 & , \text{else}\end{cases}$

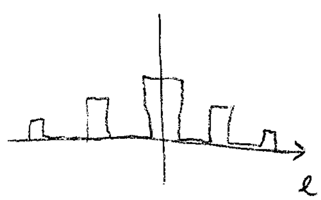
- WHAT IS 2-D RADON TRANSFORM?
- WHAT IS THE SINOGRAM?
- WHAT CHARACTERISTICS WILL THE SINOGRAM HAVE OF A CIRCULARLY SYMMETRIC FUNCTION?

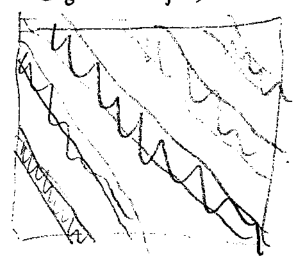
BACKPROJECTION:

SUPPOSE WE KNOW $g(\ell, \theta_0)$ FOR SOME θ_0. THERE ARE AN INFINITE # OF $f(x,y)$ THAT COULD GIVE THIS SINGLE PROJECTION. WE TAKE AN INTUITIVE GUESS AT WHAT $f(x,y)$ MIGHT LOOK LIKE (IGNORING SCALING) BY "SMEARING" THE PROJECTION IN THE PROJECTION DIRECTION TO FORM AN IMAGE:

$$\boxed{b_{\theta_0}(x,y) = g(\underbrace{x\cos\theta_0 + y\sin\theta_0}_{x\cos\theta_0 + y\sin\theta_0 = \ell}, \theta_0)}$$

⇐ BACK PROJECTION AT ANGLE θ_0 FOR $g(\ell, \theta_0)$





- IF WE SUM $b_\theta(x,y)$ AT A BUNCH OF DIFFERENT θ, WE START TO GET AN APPROXIMATION TO THE IMAGE.
- IN THE CONTINUOUS CASE, WE HAVE:

$$\boxed{f_b(x,y) = \int_0^\pi b_\theta(x,y)\, d\theta}$$

⇐ "LAMINOGRAM" OR "BACK PROJECTION SUMMATION IMAGE"

- THIS IS WHAT EARLY SCANNERS DID, BUT IT IS ONLY AN APPROXIMATION. (IN OTHER WORDS, IT IS WRONG.)

PROJECTION - SLICE THEOREM:

CONSIDER A PROJECTION $g(\ell, \theta_0)$ OF OUR OBJECT $f(x,y)$.

IF $G(k_\ell, \theta_0)$ IS THE 1-D FOURIER TRANSFORM (WITH RESPECT TO ℓ) OF $g(\ell, \theta_0)$, THEN:

$$\boxed{G(k_\ell, \theta_0) = F(k_\ell \cos\theta, k_\ell \sin\theta)}$$

"PROJECTION - SLICE THEOREM" OR "CENTRAL SECTION THEOREM"

WHERE $F(k_x, k_y)$ IS THE 2D FOURIER TRANSFORM OF $f(x,y)$.

PROOF:

$$G(k_\ell, \theta_0) = \int_{-\infty}^{\infty} g(\ell, \theta) e^{-j2\pi k_\ell \ell} d\ell$$

$\Leftarrow$ DEFINITION OF 1D F.T.

$$= \int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta_0 + y\sin\theta_0 - \ell)\,dx\,dy \right]}_{g(\ell,\theta_0)} e^{-j2\pi k_\ell \ell} d\ell$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) \int_{-\infty}^{\infty} e^{-j2\pi k_\ell \ell} \delta(x\cos\theta_0 + y\sin\theta_0 - \ell)\, d\ell\, dx\, dy$$

SIFTING PROPERTY OF DELTA

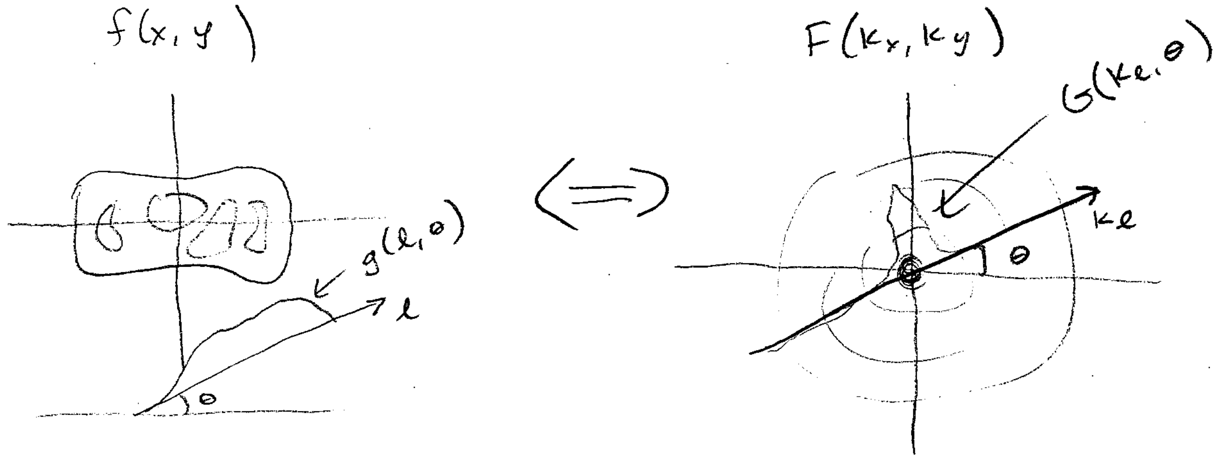
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi k_\ell (x\cos\theta_0 + y\sin\theta_0)}\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi (k_\ell \cos\theta_0\, x + k_\ell \sin\theta_0\, y)}\, dx\, dy$$

$$= F(k_\ell \cos\theta_0, k_\ell \sin\theta_0)$$

✓

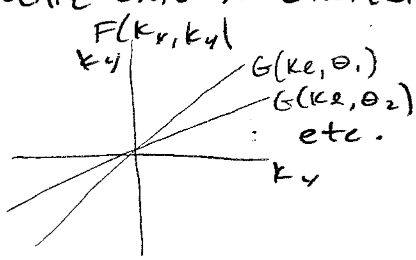
GRAPHICALLY THIS MEANS:



- THE PROJECTION-SLICE THEOREM FORMS THE BASIS FOR MODERN CT RECONSTRUCTION TECHNIQUES.
- ALSO HAS APPLICATIONS IN MRI!

THE FOURIER METHOD OF CT RECONSTRUCTION:

- MEASURE PROJECTIONS $g(\ell,\theta)$ AT A BUNCH OF θ.
- 1-D FOURIER TRANSFORM TO GET $G(k_\ell,\theta)$.
- INTERPOLATE ONTO A CARTESIAN GRID TO GET $F(k_x,k_y)$



- WE NOW HAVE $F(k_x,k_y)$ AND CAN DO AN INVERSE 2DFT TO GET $f(x,y)$!

– THIS IS VERY ACCURATE, BUT COMPUTATIONALLY INTENSIVE.

– CAN WE USE PROJECTION-SLICE THEOREM TO IMPROVE BACK PROJECTION? YES! $\Longrightarrow$ FILTERED BACK PROJECTION

FILTERED BACKPROJECTION:

RECALL THAT:

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(k_x,k_y)\, e^{j2\pi(x k_x + y k_y)}\, dk_x\, dk_y$$

⇐ JUST DEFINITION OF 2DFT INVERSE

WE CAN WRITE THIS IN POLAR COORDINATES AS:

$$f(x,y) = \int_0^{2\pi}\int_0^{\infty} F(k_\ell \cos\theta, k_\ell \sin\theta)\, e^{j2\pi(k_\ell\cos\theta\, x + k_\ell \sin\theta\, y)}\, k_\ell\, dk_\ell\, d\theta$$

SINCE: $k_x = k_\ell \cos\theta$

$k_y = k_\ell \sin\theta$

FROM THE PROJECTION-SLICE THEOREM, WE KNOW:

$$F(k_\ell\cos\theta, k_\ell\sin\theta) = G(k_\ell, \theta)$$

SO:

$$f(x,y) = \int_0^{2\pi}\int_0^{\infty} G(k_\ell,\theta)\, e^{j2\pi k_\ell(x\cos\theta + y\sin\theta)}\, k_\ell\, dk_\ell\, d\theta$$

USING: $g(\ell,\theta) = g(-\ell, \theta+\pi)$, YOU FIND:

$$f(x,y) = \int_0^{\pi}\int_{-\infty}^{\infty} |k_\ell|\, G(k_\ell,\theta)\, e^{j2\pi k_\ell(x\cos\theta + y\sin\theta)}\, dk_\ell\, d\theta$$

OR:

$$f(x,y) = \int_0^{\pi}\left[\int_{-\infty}^{\infty} |k_\ell|\, G(k_\ell,\theta)\, e^{j2\pi k_\ell \ell}\, dk_\ell\right]_{\ell = x\cos\theta + y\sin\theta} d\theta$$

THIS IS A FILTERED VERSION OF $g(\ell,\theta)$!
(MULTIPLICATION IN FOURIER SPACE)

SO IN FILTERED BACKPROJECTION, WE FILTER EACH $g(\ell,\theta)$ [BY TAKING 1-D F.T. AND MULTIPLYING BY FILTER TRANSFER FUNCTION $|k_\ell|$, AND THEN DOING INVERSE 1-D F.T.] AND THEN BACKPROJECT!

INTUITION:

- COMPENSATION FOR SAMPLING DENSITY

CONVOLUTION BACKPROJECTION:

- SAME THING AS FILTERED BACKPROJECTION, EXCEPT FILTERING OF $g(\ell, \theta)$ IS DONE BY CONVOLUTION IN THE SPATIAL DOMAIN W/ $\mathcal{F}^{-1}\left[|k_\ell|\right]$

"RAMP FILTER"

FAN-BEAM GEOMETRY:

- WE WON'T GO INTO TOO MUCH DETAIL. IN BOOK
- COMMON VARIATION FOR FAN-BEAM GEOMETRY IS CONVOLUTION WEIGHTED-BACKPROJECTION
  - FILTERED PROJECTIONS ARE BACKPROJECTED ALONG THE X-RAY PATHS OF INTEGRATION.
  - WEIGHTING IS APPLIED THAT DEPENDS ON DISTANCE FROM SOURCE.

A FEW WORDS ON SAMPLING IN CT:

- IN OUR DISCUSSIONS, WE HAVE ASSUMED CONTINUOUS DATA IN BOTH θ AND ℓ.
- IN PRACTICE, WE HAVE A LIMITED NUMBER OF PROJECTIONS (CALLED "VIEWS") (DIFFERENT VALUES OF θ) AND DETECTOR SAMPLES (DIFFERENT VALUES OF ℓ)
- HOW MANY VIEWS & DETECTOR SAMPLES ARE NEEDED? (EASIEST TO DERIVE FOR 1G GEOMETRIES)

$\boxed{\text{WE WILL DISCUSS THIS IN THE ADVANCED LECTURE MONDAY.}}$