DEVELOPMENT OF THE SIGNAL EQUATION:

- WE WANT TO DEVELOP A MATHEMATICAL DESCRIPTION OF THE MR SIGNAL FROM A 3D SAMPLE TO BE IMAGED.

- ASSUME A UNIFORMLY SENSITIVE RECEIVER COIL OVER THE ENTIRE F.O.V. (BAD ASSUMPTION, BUT OKAY FOR NOW.)

- RECEIVER COIL IS DESIGNED TO DETECT FLUX CHANGES IN TRANSVERSE DIRECTION (M_{xy}, SINCE WE HAVE ORIENTED $\vec{B}_0$ IN THE z DIRECTION)

SIGNAL IS THE SUM OF ALL PRECESSING MAGNETIZATIONS IN THE VOLUME. SIMPLISTICALLY:

$$s_r(t) = \int_{\text{VOLUME}} M_{xy}(\vec{r},t)\,dV = \int_x\int_y\int_z M_{xy}(x,y,z,t)\,dx\,dy\,dz$$

HOW ARE WE GOING TO REPRESENT $M_{xy}(\vec{r},t)$?

PHASOR NOTATION:

$$M_{xy}(\vec{r},t) = \underbrace{M^\circ_{xy}(\vec{r})}_{\text{INITIAL CONDITION } \cancel{\text{AFTER TIP}}}\ \underbrace{e^{-t/T_2(\vec{r})}}_{T_2 \text{ DECAY TERM}}\ \underbrace{e^{-i\omega_0 t}}_{\text{PRECESSION TERM (PHASE)}} \Rightarrow M_{xy} \equiv M_x + iM_y$$

WHERE:

$$\boxed{\begin{aligned} M^\circ_{xy}(\vec{r}) &= M^\circ_x + iM^\circ_y \\ \omega_0 &= \gamma B_0 \end{aligned}}$$

WHAT HAPPENS WHEN WE INCLUDE THE EFFECTS FROM GRADIENTS?

- WE SPEED UP OR SLOW DOWN THE PRECESSION (PHASE) TERM!

- IN GENERAL:

$$\boxed{B(\vec{r},t) = \left(B_0 + G_x^{(t)}x + G_y^{(t)}y + G_z^{(t)}z\right)\hat{k} = \left(B_0 + \vec{G}(t)\cdot\vec{r}\right)\hat{k}}$$

SO:

$$\boxed{\omega(\vec{r},t) = \gamma B_0 + \gamma\vec{G}(t)\cdot\vec{r}} = \boxed{\omega_0 + \gamma\vec{G}(t)\cdot\vec{r}}$$

NOW, CAN I SIMPLY PLUG THIS EXPRESSION IN TO MY EXPRESSION FOR $M_{xy}(\vec{r},t)$??

NO!

PHASE TERM

MY PHASOR TERM $e^{-i\boxed{\omega_0 t}}$ GIVES THE PHASE AT TIME t. THIS DEPENDS NOT ONLY ON THE CURRENT FREQUENCY $\omega(t)$ BUT ON FREQUENCIES IN THE PAST!

IN FACT:

$$\text{PHASE} = \int_0^t \omega(\vec{r},\tau)\,d\tau$$

IN THE CASE WHERE:

$$\omega(\vec{r},t) = \omega_0$$

WE HAVE:

$$\text{PHASE} = \int_0^t \omega_0\,d\tau = \boxed{\omega_0 t}$$

NOW, WHEN $\omega(\vec{r},t)$ IS TIME VARYING, WE HAVE:

$$\text{PHASE} = \int_0^t \omega(\vec{r},\tau)\,d\tau = \int_0^t \left[\omega_0 + \gamma \vec{G}(\tau)\cdot\vec{r}\right]d\tau$$

$$\text{PHASE} = \int_0^t \omega_0\,d\tau + \int_0^t \gamma \vec{G}(\tau)\cdot\vec{r}\,d\tau$$

$$\text{PHASE} = \omega_0 t + \int_0^t \gamma \vec{G}(\tau)\cdot\vec{r}\,d\tau$$

WE CAN PLUG THIS IN WHERE WE SIMPLY HAD $\underline{\omega_0 t}$ BEFORE:

$$M_{xy}(\vec{r},t) = M^{\circ}_{xy}(\vec{r})\,e^{-t/T_2(\vec{r})}\,e^{-i\left[\omega_0 t + \int_0^t \gamma \vec{G}(\tau)\cdot\vec{r}\,d\tau\right]}$$

$$\boxed{M_{xy}(\vec{r},t) = M^{\circ}_{xy}(\vec{r})\,e^{-t/T_2(\vec{r})}\,e^{-i\omega_0 t}\,e^{-i\gamma\int_0^t \vec{G}(\tau)\cdot\vec{r}\,d\tau}}$$

PLUGGING INTO OUR SIGNAL EQUATION, WE HAVE:

$$S_r(t) = \iiint_{x\,y\,z} M^\circ_{xy}(x,y,z)\, \underbrace{e^{-t/T_2(\vec{r})}}\, e^{-i\omega_0 t}\, e^{-i\gamma\int_0^t \vec{G}(\tau)\cdot\vec{r}\,d\tau}\, dx\,dy\,dz$$

$\Rightarrow$ TO SIMPLIFY, WE'RE GOING TO IGNORE RELAXATION FOR NOW.

$\Rightarrow$ WE ALSO ASSUME SLICE SELECTIVE EXCITATION TO GO TO 2D (FROM 3D)

- IF WE TIP THE WHOLE VOLUME UNIFORMLY, THEN:

$$m(x,y) \propto M^\circ_{xy}(x,y)$$ $\Leftarrow$ WITH z INTEGRATED OVER SLICE WIDTH

THE THING WE'RE IMAGING

SO:

$$S_r(t) = \iint_{x\,y} m(x,y)\, e^{-i\omega_0 t}\, e^{-i\gamma\int_0^t \vec{G}(\tau)\cdot\vec{r}\,d\tau}\, dx\,dy$$

NOW LET'S DEMODULATE OUR SIGNAL AT FREQ. ω_0:

$$s(t) = S_r(t)\, e^{+i\omega_0 t}$$

THEN:

$$s(t) = \int_x \int_y m(x,y)\, e^{-i\gamma\left[\int_0^t x G_x(\tau)\,d\tau + \int_0^t y G_y(\tau)\,d\tau\right]}\, dx\,dy$$

PUTTING IT ALL TOGETHER, WE CAN WRITE:

$$\boxed{s(t) = \int_x \int_y m(x,y)\, e^{-i2\pi\left[k_x(t)\,x + k_y(t)\,y\right]}\,dx\,dy}$$

WHERE: $$\boxed{\begin{aligned} k_x(t) &= \frac{\gamma}{2\pi}\int_0^t G_x(\tau)\,d\tau \\ k_y(t) &= \frac{\gamma}{2\pi}\int_0^t G_y(\tau)\,d\tau \end{aligned}}$$

NOTICE THAT $s(t)$ LOOKS A LOT LIKE A 2D FOURIER TRANSFORM!!

IF WE LOOK AT THE F.T. OF $m(x,y)$:

$$M(k_x,k_y) = \int_x \int_y m(x,y)\, e^{-i2\pi(k_x x + k_y y)}\,dx\,dy$$

THEN WE SEE THAT:

$$s(t) = M\big(k_x(t), k_y(t)\big)$$

IN OTHER WORDS, $s(t)$ IS GIVING US THE VALUE OF THE 2DFT OF $m(x,y)$ AT K-SPACE POSITION $[k_x(t), k_y(t)]$ AT A GIVEN VALUE OF t!

THIS IS THE BASIS FOR THE VERY POWERFUL "FOURIER INTERPRETATION" OF THE SIGNAL EQUATION.

BY MANIPULATING OUR TIME-VARYING GRADIENTS $G_x(t)$ AND $G_y(t)$ APPROPRIATELY, WE CAN "TRAVERSE" K-SPACE IN ANY WAY WE WANT, AND SAMPLE VALUES OF $M(k_x, k_y)$ FROM THE SIGNAL $s(t)$.

THIS IS VERY POWERFUL, AND THE BASIS FOR MRI PULSE SEQUENCE DESIGN.